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Bi-local baryon interpolating fields with two flavours

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Abstract. We construct bi-local interpolating field operators for baryons consisting of three quarks with two flavors, assuming good isospin symmetry. We use the restrictions following from the Pauli principle to derive relations/identities among the baryon operators with identical quantum numbers. Such relations that follow from the combined spatial, Dirac, color, and isospin Fierz transformations may be called the (total/complete) Fierz identities. These relations reduce the number of independent baryon operators with any given spin and isospin. We also study the Abelian and non-Abelian chiral transformation properties of these fields and place them into baryon chiral multiplets. Thus we derive the independent baryon interpolating fields with given values of spin (Lorentz group representation), chiral symmetry ($U_L(2) \times U_R(2)$ group representation) and isospin appropriate for the first angular excited states of the nucleon.

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1 Introduction

QCD is at present our best theoretical framework for the description of hadrons and the chiral symmetry is one of its global symmetries that plays a key role in hadron physics. Interpolating fields of hadrons and of baryons in particular have been part and parcel of lattice QCD and QCD sum rule calculations for almost three decades. Many such studies suggest that the minimal local structure, i.e. three quark fields without derivatives, baryon operators successfully describe properties of the lowest-lying baryon ground state(s). Moving beyond the ground states to describe even the lowest-lying excited states turns out to be a challenge in the local operator approximation, however. Interpolators B for baryons with spin larger than 3/2 consisting of three quarks cannot be local operators in the continuum limit, see Ref. [1].

Indeed, for baryons with total angular momenta larger than 3/2, the "orbital" angular momentum contribution must be non-zero, that can only be introduced by means of an additional four-vector, see Ref. [1]. With local operators, there is only one such four-vector: the four-derivative, or equivalently the four-momentum of the baryon. Manifestly, an application of the four-momentum operator to the baryon field cannot change its angular momentum. One therefore needs another four-vector to "excite" the orbital angular momentum, and the two independent separations between the quarks (the two Jacobi relative coordinate four-vectors) are precisely what one needs. In plain English, one needs to have at least two quark fields at two different locations, that leads to a non-local baryon field.

In this paper we address the question of bi-local baryon fields in the continuum limit, as the first and simplest extension beyond the local approximation. The properties of interpolating fields, such as their Fierz identities [2,3,4] and chiral properties [5,6,7] have been explored in detail only in the lower spin and local operator limit, whereas the study of higher spin fields has only recently begun, and that exclusively on the discrete space-time lattice(s), see e.g. Ref. [8]; the higher spin fields in the continuum space-time have not been dealt with, as yet, to our knowledge.

We shall construct bi-local baryon interpolating fields in such a way that they belong to reducible representations of the Lorentz group described by two (half-)integers (p, q) and to irreducible representations of the isospin $SU(2)$ group, described by the isospin, a (half-)integer I, where the quark field is expressed as the iso-doublet field. It was not *a priori* obvious, however, that they also belong to the same (irreducible) representations of the chiral group $SU(2)_R \times SU(2)_L$, where $I_{R,L}$ label the representations of the right-, and left- isospin groups $SU(2)_{R,L}$. The "ordinary" (vector) isospin I is the quantum mechanical sum of the right- and left- isospins: $I = |I_L - I_R|, \ldots, I_L + I_R$. We have shown in Ref. [9,

10, 11] that the Fierz identities among local baryon operators also determine their chiral multiplet structure and shall show here that the same holds for bi-local fields.

This should not be surprising as the Fierz identities form an implementation of the Pauli principle, and different permutation symmetry classes form distinct multiplets of composite particles. Hence it is necessary to carefully take into account the Fierz identities also among bi-local baryon operators. The two Jacobi relative coordinates form the basis of the two-dimensional irreducible representation of the permutation group S_3, which fact leads to generalized/composite Fierz identities and further simplifies the classification of the resulting bi-local fields under the Pauli principle.

The standard isospin formalism greatly facilitates derivation of the Fierz identities and chiral transformations of baryon operators, due to the fact that both the quarks and the nucleons belong to the iso-doublet representation. The composite Fierz identities (i.e. in the spatial, Dirac, isospin and color space) and the chiral transformations of baryons are then straightforwardly derived using the iso-doublet representation.

We give an explicit derivation of these identities for two reasons: a) this is the first such derivation, to our knowledge; and b) because of its relative simplicity, we hope that it will show the way to the chiral $SU(3)_R \times SU(3)_L$ extension, that is (substantially) more complicated, and encourage others to attack this and the tri-local field problem. This framework can be applied to other extensions, such as the inclusion of multi-quark configurations, and/or of gluon fields into the baryon interpolators.

This paper is organized as follows. In section 2, we firstly define all possible bi-local baryon operators. We classify the baryon operators according to the representations of the Lorentz and the isospin groups. Then we apply the Fierz transformation to obtain Fierz identities among the baryon operators for each representation of the Lorentz and chiral isospin group. In section 3, we derive the Abelian and non-Abelian chiral transformations of the baryon operators as functions of the quarks' chiral transformation parameters, using the iso-doublet representation. All possible chiral multiplets for the bi-local baryon operators are displayed by taking into account the Fierz identities. The final section is a summary and an outlook to possible future extensions and applications. In Appendix A we define all possible quark bi-linear fields with at most one derivative and summarize their chiral transformations. In Appendix B we define the Fierz transformations in the color, flavor and spatial spaces.

2 Baryon Field Operators

We start with some general comments about three-quark baryon interpolating operators. An interpolating operator B for baryons consisting of three quarks cannot be local in general: Indeed, for baryons with total angular momenta larger than $3/2$, the "orbital" angular momentum contribution must be non-zero, and that can only be described by means of additional four-vector operators. With local operators, there is only one such four-vector: the four-derivative, or equivalently the four-momentum of the baryon. Manifestly, application of this operator to the baryon field cannot change its angular momentum. One, therefore needs another four-vector to "excite" the orbital angular momentum, and the separations between the quarks (two Jacobi relative coordinates) are precisely what one needs.

So, in general one must write[1]

$$B(x, y, z) \sim \epsilon_{abc} \left(q_a^T(x) \Gamma_1 q_b(y) \right) \Gamma_2 q_c(z), \tag{1}$$

where $q(x) = (u(x),\ d(x))^T$ is an iso-doublet quark field at location x, the superscript T represents the transpose and the indices a, b and c represent the color. Here the antisymmetric tensor in color space ϵ_{abc} ensures the baryons' being color singlets. From now on, we shall omit the color indices always assuming that the system is a color singlet, which further implies that any pair of quarks (a "diquark") is in a colour anti-triplet state.

The matrices $\Gamma_{1,2}$ are tensor products of Dirac and isospin matrices. With a suitable choice of $\Gamma_{1,2}$, the baryon operators are defined so that they form an irreducible representation of the Lorentz and isospin groups, as we shall show in this section.

Note that we use the iso-doublet form for the quark field q, although the explicit expressions in terms of up and down quarks are usually employed in lattice QCD and QCD sum rule studies. We have shown in Ref. [11] that the iso-doublet formulation leads to a simple classification of baryons into isospin multiplets and to a straightforward derivation of Fierz identities and chiral transformations of baryon operators.

As the tri-local fields are substantially more complicated than the bi-local ones, and neither have been considered in the literature, as yet, we shall proceed with an analysis of the latter.

[1] Of course one must include the color-dependent and path-dependent "gauge factors". We shall drop them henceforth, to keep the notation simple.

2.1 Bi-local Baryon Fields

A bi-local interpolating operator $B(x,y)$ for baryons consisting of three quarks can be generally written as

$$\begin{aligned} B(x,y) &\sim \left(q^T(x)\Gamma_1 q(y)\right)\Gamma_2 q(x) + \left(q^T(x)\Gamma_3 q(x)\right)\Gamma_4 q(y) \\ &= D^i(x,y)\Gamma^i q(x) + D^j(x,x)\Gamma^j q(y), \end{aligned} \tag{2}$$

where $D^i(x,y)$ are bi-local diquark fields at location x,y, see Appendix A. The Pauli principle relates the two terms in Eq. (2). Here we shall consider the Pauli principle in two steps.

The first step is to apply the Pauli principle to the first and second quarks, i.e. to the diquarks, as discussed in Appendix A. Second, an additional constraint comes from the permutation of the second and the third quark, which corresponds to the usual Fierz transformation.

Note that the Fierz transformation connects only baryon operators belonging to the same Lorentz and isospin group multiplets. We may, therefore, classify the baryon operators according to their Lorentz and isospin representations following Chung et al [3]. It has been known that such baryon operators may couple either to the even- or to the odd-parity states. In the following discussion all the baryon operators will be defined as having even parity. We note, however, that two different isospin baryon operators belonging to the same chiral multiplet may have opposite parities.

To construct the bi-local baryon fields, we follow the same approach we used before, and we classify them according to their spin and isospin. It is convenient to introduce a "tilde-transposed" quark field $\tilde{q}$ as follows

$$\tilde{q} = q^T C\gamma_5(i\tau_2), \tag{3}$$

where $C = i\gamma_2\gamma_0$ is the Dirac field charge conjugation operator, τ_2 is the second isospin Pauli matrix, whose elements form the antisymmetric tensor in isodoublet space.

2.2 $J = \frac{1}{2}$ and $I = \frac{1}{2}$ fields

Firstly, we consider the simplest case $D(\frac{1}{2},0)_{I=\frac{1}{2}}$, where $D(\frac{1}{2},0)$ denotes the representation of the Lorentz group and $I = \frac{1}{2}$ denotes the isospin. There are twenty bi-local nucleon operators of $J = \frac{1}{2}$ and $I = \frac{1}{2}$

$$\begin{aligned}
&\begin{cases} N_1(x,y) = (\tilde{q}(x)q(y))q(x)\,, \\ N_2(x,y) = (\tilde{q}(x)\gamma_5 q(y))\gamma_5 q(x)\,, \\ N_3(x,y) = (\tilde{q}(x)\gamma_\mu q(y))\gamma^\mu q(x)\,, \\ N_4(x,y) = (\tilde{q}(x)\gamma_\mu\gamma_5\tau^i q(y))\gamma^\mu\gamma_5\tau^i q(x)\,, \\ N_5(x,y) = (\tilde{q}(x)\sigma_{\mu\nu}\tau^i q(y))\sigma^{\mu\nu}\tau^i q(x)\,, \end{cases} \\
&\begin{cases} N_6(x,y) = (\tilde{q}(x)\tau^i q(y))\tau^i q(x)\,, \\ N_7(x,y) = (\tilde{q}(x)\gamma_5\tau^i q(y))\gamma_5\tau^i q(x)\,, \\ N_8(x,y) = (\tilde{q}(x)\gamma_\mu\tau^i q(y))\gamma^\mu\tau^i q(x)\,, \\ N_9(x,y) = (\tilde{q}(x)\gamma_\mu\gamma_5 q(y))\gamma^\mu\gamma_5 q(x)\,, \\ N_{10}(x,y) = (\tilde{q}(x)\sigma_{\mu\nu} q(y))\sigma^{\mu\nu} q(x)\,, \end{cases} \\
&\begin{cases} N_{11}(x,y) = (\tilde{q}(x)q(x))q(y)\,, \\ N_{12}(x,y) = (\tilde{q}(x)\gamma_5 q(x))\gamma_5 q(y)\,, \\ N_{13}(x,y) = (\tilde{q}(x)\gamma_\mu q(x))\gamma^\mu q(y)\,, \\ N_{14}(x,y) = (\tilde{q}(x)\gamma_\mu\gamma_5\tau^i q(x))\gamma^\mu\gamma_5\tau^i q(y)\,, \\ N_{15}(x,y) = (\tilde{q}(x)\sigma_{\mu\nu}\tau^i q(x))\sigma^{\mu\nu}\tau^i q(y)\,, \end{cases} \\
&\begin{cases} N_{16}(x,y) = (\tilde{q}(x)\tau^i q(x))\tau^i q(y)\,, \\ N_{17}(x,y) = (\tilde{q}(x)\gamma_5\tau^i q(x))\tau^i\gamma_5 q(y)\,, \\ N_{18}(x,y) = (\tilde{q}(x)\gamma_\mu\tau^i q(x))\gamma^\mu\tau^i q(y)\,, \\ N_{19}(x,y) = (\tilde{q}(x)\gamma_\mu\gamma_5 q(x))\gamma^\mu\gamma_5 q(y)\,, \\ N_{20}(x,y) = (\tilde{q}(x)\sigma_{\mu\nu} q(x))\sigma^{\mu\nu} q(y)\,. \end{cases}
\end{aligned} \tag{4}$$

Among them N_{16}-N_{20} vanish due to the Pauli principle. Using the Fierz identities for the Dirac spin and isospin indices we obtain the following identities:

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \\ N_7 \\ N_8 \\ N_9 \\ N_{10} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 1 & 1 & 1 & -1 & \frac{1}{2} & 1 & 1 & 1 & -1 & \frac{1}{2} \\ 1 & 1 & -1 & 1 & \frac{1}{2} & 1 & 1 & -1 & 1 & \frac{1}{2} \\ 4 & -4 & -2 & -2 & 0 & 4 & -4 & -2 & -2 & 0 \\ -12 & 12 & -6 & 2 & 0 & 4 & -4 & 2 & -6 & 0 \\ 36 & 36 & 0 & 0 & 2 & -12 & -12 & 0 & 0 & -6 \\ 3 & 3 & 3 & 1 & -\frac{1}{2} & -1 & -1 & -1 & -3 & \frac{3}{2} \\ 3 & 3 & -3 & -1 & -\frac{1}{2} & -1 & -1 & 1 & 3 & \frac{3}{2} \\ 12 & -12 & -6 & 2 & 0 & -4 & 4 & 2 & -6 & 0 \\ -4 & 4 & -2 & -2 & 0 & -4 & 4 & -2 & -2 & 0 \\ 12 & 12 & 0 & 0 & -2 & 12 & 12 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} N_{11} \\ N_{12} \\ N_{13} \\ N_{14} \\ N_{15} \\ N_{16} \\ N_{17} \\ N_{18} \\ N_{19} \\ N_{20} \end{pmatrix} ,$$

and

$$\begin{pmatrix} N_{11} \\ N_{12} \\ N_{13} \\ N_{14} \\ N_{15} \\ N_{16} \\ N_{17} \\ N_{18} \\ N_{19} \\ N_{20} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 1 & 1 & 1 & -1 & \frac{1}{2} & 1 & 1 & 1 & -1 & \frac{1}{2} \\ 1 & 1 & -1 & 1 & \frac{1}{2} & 1 & 1 & -1 & 1 & \frac{1}{2} \\ 4 & -4 & -2 & -2 & 0 & 4 & -4 & -2 & -2 & 0 \\ -12 & 12 & -6 & 2 & 0 & 4 & -4 & 2 & -6 & 0 \\ 36 & 36 & 0 & 0 & 2 & -12 & -12 & 0 & 0 & -6 \\ 3 & 3 & 3 & 1 & -\frac{1}{2} & -1 & -1 & -1 & -3 & \frac{3}{2} \\ 3 & 3 & -3 & -1 & -\frac{1}{2} & -1 & -1 & 1 & 3 & \frac{3}{2} \\ 12 & -12 & -6 & 2 & 0 & -4 & 4 & 2 & -6 & 0 \\ -4 & 4 & -2 & -2 & 0 & -4 & 4 & -2 & -2 & 0 \\ 12 & 12 & 0 & 0 & -2 & 12 & 12 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \\ N_7 \\ N_8 \\ N_9 \\ N_{10} \end{pmatrix} .$$

Solving these equations, we obtain the following solutions

$$\begin{pmatrix} N_6 \\ N_7 \\ N_8 \\ N_9 \\ N_{10} \\ N_{11} \\ N_{12} \\ N_{13} \\ N_{14} \\ N_{15} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} -6 & -6 & -6 & -2 & 1 \\ -6 & -6 & 6 & 2 & 1 \\ -24 & 24 & 12 & -4 & 0 \\ 8 & -8 & 4 & 4 & 0 \\ -24 & -24 & 0 & 0 & 4 \\ -6 & 2 & 2 & -2 & 1 \\ 2 & -6 & -2 & 2 & 1 \\ 8 & -8 & -12 & -4 & 0 \\ -24 & 24 & -12 & -4 & 0 \\ 72 & 72 & 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \end{pmatrix} . \tag{5}$$

Therefore, only these five $(N_1, N_2, N_3, N_4, N_5)$ of the original twenty operators survive the Pauli principle.

2.3 $J = \frac{1}{2}$ and $I = \frac{3}{2}$ fields

Next we consider $D(\frac{1}{2}, 0)_{I=\frac{3}{2}}$ fields. Baryon operators with $I = \frac{3}{2}$ must contain either the axial-vector or the tensor diquark, so there are ten bi-local baryon fields with $J = \frac{1}{2}$ and $I = \frac{3}{2}$ left

$$\begin{aligned} &\begin{cases} \Delta^i_4(x,y) = (\tilde{q}(x)\gamma_\mu\gamma_5\tau^j q(y))\gamma^\mu\gamma_5 P^{ij}_{3/2} q(x) \,, \\ \Delta^i_5(x,y) = (\tilde{q}(x)\sigma_{\mu\nu}\tau^j q(y))\sigma^{\mu\nu} P^{ij}_{3/2} q(x) \,, \end{cases} \\ &\begin{cases} \Delta^i_6(x,y) = (\tilde{q}(x)\tau^j q(y)) P^{ij}_{3/2} q(x) \,, \\ \Delta^i_7(x,y) = (\tilde{q}(x)\gamma_5\tau^j q(y))\gamma_5 P^{ij}_{3/2} q(x) \,, \\ \Delta^i_8(x,y) = (\tilde{q}(x)\gamma_\mu\tau^j q(y))\gamma^\mu P^{ij}_{3/2} q(x) \,, \end{cases} \\ &\begin{cases} \Delta^i_{14}(x,y) = (\tilde{q}(x)\gamma_\mu\gamma_5\tau^j q(x))\gamma^\mu\gamma_5 P^{ij}_{3/2} q(y) \,, \\ \Delta^i_{15}(x,y) = (\tilde{q}(x)\sigma_{\mu\nu}\tau^j q(x))\sigma^{\mu\nu} P^{ij}_{3/2} q(y) \,, \end{cases} \\ &\begin{cases} \Delta^i_{16}(x,y) = (\tilde{q}(x)\tau^j q(x)) P^{ij}_{3/2} q(y) \,, \\ \Delta^i_{17}(x,y) = (\tilde{q}(x)\gamma_5\tau^j q(x))\gamma_5 P^{ij}_{3/2} q(y) \,, \\ \Delta^i_{18}(x,y) = (\tilde{q}(x)\gamma_\mu\tau^j q(x))\gamma^\mu P^{ij}_{3/2} q(y) \,. \end{cases} \end{aligned} \tag{6}$$

Among these operators Δ^i_{16}-Δ^i_{18} vanishes due to the Pauli principle. Here $P^{ij}_{3/2}$ is the isospin-projection operator for $I = \frac{3}{2}$, which is defined, together with an isospin-projection operator $P^{ij}_{1/2}$ for $I = \frac{1}{2}$, as

$$P^{ij}_{3/2} = \delta^{ij} - \frac{1}{3}\tau^i\tau^j\,,\; P^{ij}_{1/2} = \frac{1}{3}\tau^i\tau^j\,. \tag{7}$$

The $I = \frac{3}{2}$ projection operator satisfies $\tau^i P^{ij}_{\frac{3}{2}} = 0$, which ensures $\tau^i\Delta^i_{4,5} = 0$. Again the Fierz transformation provides the following relations

$$\begin{pmatrix} \Delta^i_6 \\ \Delta^i_7 \\ \Delta^i_8 \\ \Delta^i_4 \\ \Delta^i_5 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & -1 & \frac{1}{2} \\ 1 & 1 & -1 & 1 & \frac{1}{2} \\ 4 & -4 & -2 & -2 & 0 \\ -4 & 4 & -2 & -2 & 0 \\ 12 & 12 & 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} \Delta^i_{16} \\ \Delta^i_{17} \\ \Delta^i_{18} \\ \Delta^i_{14} \\ \Delta^i_{15} \end{pmatrix}\,, \tag{8}$$

and

$$\begin{pmatrix} \Delta^i_{16} \\ \Delta^i_{17} \\ \Delta^i_{18} \\ \Delta^i_{14} \\ \Delta^i_{15} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & -1 & \frac{1}{2} \\ 1 & 1 & -1 & 1 & \frac{1}{2} \\ 4 & -4 & -2 & -2 & 0 \\ -4 & 4 & -2 & -2 & 0 \\ 12 & 12 & 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} \Delta^i_6 \\ \Delta^i_7 \\ \Delta^i_8 \\ \Delta^i_4 \\ \Delta^i_5 \end{pmatrix}\,. \tag{9}$$

Solving these equations, we obtain the following solutions

$$\begin{pmatrix} \Delta^i_6 \\ \Delta^i_7 \\ \Delta^i_8 \\ \Delta^i_{14} \\ \Delta^i_{15} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ -2 & -1 \\ 4 & 0 \\ -8 & 0 \\ 0 & -8 \end{pmatrix}\begin{pmatrix} \Delta^i_4 \\ \Delta^i_5 \end{pmatrix}\,. \tag{10}$$

Therefore, only two (Δ^i_4, Δ^i_5) of the ten Δ operators are independent under the Pauli principle.

2.4 $J = \frac{3}{2}$ and $I = \frac{1}{2}$ fields

There are two possible fields/Lorentz group representations with $J = \frac{3}{2}$: 1) the $D(1, \frac{1}{2})$ and 2) the $D(\frac{3}{2}, 0)$.

2.4.1 $D(1, \frac{1}{2})$ and $I = \frac{1}{2}$

For $J = \frac{3}{2}$ fields one of the allowed Lorentz representations is $D(1, \frac{1}{2})$. In this case, baryon operators may contain the vector and the axial-vector, or the tensor diquark. So we altogether have twelve bi-local baryon fields

$$\left\{\begin{array}{l} N_{3\mu}(x,y) = (\tilde{q}(x)\gamma_\nu q(y))\Gamma^{\mu\nu}_{3/2}\gamma_5 q(x)\,, \\ N_{4\mu}(x,y) = (\tilde{q}(x)\gamma_\nu\gamma_5\tau^i q(y))\Gamma^{\mu\nu}_{3/2}\tau^i q(x)\,, \\ N_{5\mu}(x,y) = (\tilde{q}(x)\sigma_{\alpha\beta}\tau^i q(y))\Gamma^{\mu\alpha}_{3/2}\gamma^\beta\gamma_5\tau^i q(x)\,, \end{array}\right.$$

$$\left\{\begin{array}{l} N_{8\mu}(x,y) = (\tilde{q}(x)\gamma_\nu\tau^i q(y))\Gamma^{\mu\nu}_{3/2}\gamma_5\tau^i q(x)\,, \\ N_{9\mu}(x,y) = (\tilde{q}(x)\gamma_\nu\gamma_5 q(y))\Gamma^{\mu\nu}_{3/2} q(x)\,, \\ N_{10\mu}(x,y) = (\tilde{q}(x)\sigma_{\alpha\beta} q(y))\Gamma^{\mu\alpha}_{3/2}\gamma^\beta\gamma_5 q(x)\,, \end{array}\right.$$

$$\left\{\begin{array}{l} N_{13\mu}(x,y) = (\tilde{q}(x)\gamma_\nu q(x))\Gamma^{\mu\nu}_{3/2}\gamma_5 q(y)\,, \\ N_{14\mu}(x,y) = (\tilde{q}(x)\gamma_\nu\gamma_5\tau^i q(x))\Gamma^{\mu\nu}_{3/2}\tau^i q(y)\,, \\ N_{15\mu}(x,y) = (\tilde{q}(x)\sigma_{\alpha\beta}\tau^i q(x))\Gamma^{\mu\alpha}_{3/2}\gamma^\beta\gamma_5\tau^i q(y)\,, \end{array}\right.$$

$$\left\{\begin{array}{l} N_{18\mu}(x,y) = (\tilde{q}(x)\gamma_\nu\tau^i q(x))\Gamma^{\mu\nu}_{3/2}\gamma_5\tau^i q(y)\,, \\ N_{19\mu}(x,y) = (\tilde{q}(x)\gamma_\nu\gamma_5 q(x))\Gamma^{\mu\nu}_{3/2} q(y)\,, \\ N_{20\mu}(x,y) = (\tilde{q}(x)\sigma_{\alpha\beta} q(x))\Gamma^{\mu\alpha}_{3/2}\gamma^\beta\gamma_5 q(y)\,. \end{array}\right.$$

Among them $N_{18\mu}$-$N_{20\mu}$ vanishes due to the Pauli principle. Similarly to the isospin projection operators, $\Gamma^{\mu\nu}_{3/2}$ is the spin-projection operator for $J = \frac{3}{2}$ states, which is defined, together with the $J = \frac{1}{2}$ projection operator $\Gamma^{\mu\nu}_{1/2}$, by

$$\Gamma^{\mu\nu}_{3/2} = g^{\mu\nu} - \frac{1}{4}\gamma^\mu\gamma^\nu,\ \Gamma^{\mu\nu}_{1/2} = \frac{1}{4}\gamma^\mu\gamma^\nu. \tag{11}$$

Owing to this projection operator, the $J = \frac{3}{2}$ baryon operators satisfy the Rarita-Schwinger condition $\gamma_\mu N^\mu_{3,4,5} = 0$. The Fierz transformation provides the following relations

$$\begin{pmatrix} N_{3\mu} \\ N_{4\mu} \\ N_{5\mu} \\ N_{8\mu} \\ N_{9\mu} \\ N_{10\mu} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & -1 & 1 & -1 & 3 & -3 \\ 6 & 2 & 0 & -2 & -6 & 0 \\ 3 & -1 & -1 & -1 & 3 & 3 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 2 & -2 & 0 & 2 & -2 & 0 \end{pmatrix}\begin{pmatrix} N_{13\mu} \\ N_{14\mu} \\ N_{15\mu} \\ N_{18\mu} \\ N_{19\mu} \\ N_{20\mu} \end{pmatrix}, \tag{12}$$

and

$$\begin{pmatrix} N_{13\mu} \\ N_{14\mu} \\ N_{15\mu} \\ N_{18\mu} \\ N_{19\mu} \\ N_{20\mu} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & -1 & 1 & -1 & 3 & -3 \\ 6 & 2 & 0 & -2 & -6 & 0 \\ 3 & -1 & -1 & -1 & 3 & 3 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 2 & -2 & 0 & 2 & -2 & 0 \end{pmatrix}\begin{pmatrix} N_{3\mu} \\ N_{4\mu} \\ N_{5\mu} \\ N_{8\mu} \\ N_{9\mu} \\ N_{10\mu} \end{pmatrix}. \tag{13}$$

Solving these equations, we obtain the following linear relations

$$\begin{pmatrix} N_{8\mu} \\ N_{9\mu} \\ N_{10\mu} \\ N_{13\mu} \\ N_{14\mu} \\ N_{15\mu} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -3 & 1 & 1 \\ -1 & -1 & 1 \\ -2 & 2 & 0 \\ -1 & 1 & 1 \\ 3 & -3 & 1 \\ 6 & 2 & -2 \end{pmatrix}\begin{pmatrix} N_{3\mu} \\ N_{4\mu} \\ N_{5\mu} \end{pmatrix}. \tag{14}$$

Thus, we take $(N_{3\mu}, N_{4\mu}, N_{5\mu})$ as the independent fields.

2.4.2 $D(\frac{3}{2}, 0)$ and $I = \frac{1}{2}$

There are four other fields with $J = \frac{3}{2}$ and $I = \frac{1}{2}$ in the $D(\frac{3}{2}, 0)$ Lorentz group representation, i.e. that have two Lorentz indices

$$N_{5\mu\nu}(x,y) = (\tilde{q}(x)\sigma_{\alpha\beta}\tau^i q(y))\Gamma^{\mu\nu\alpha\beta}_{3/2}\tau^i q(x)\,, \tag{15}$$

$$N_{10\mu\nu}(x,y) = (\tilde{q}(x)\sigma_{\alpha\beta} q(y))\Gamma^{\mu\nu\alpha\beta}_{3/2} q(x)\,, \tag{16}$$

$$N_{15\mu\nu}(x,y) = (\tilde{q}(x)\sigma_{\alpha\beta}\tau^i q(x))\Gamma^{\mu\nu\alpha\beta}_{3/2}\tau^i q(y)\,, \tag{17}$$

$$N_{20\mu\nu}(x,y) = (\tilde{q}(x)\sigma_{\alpha\beta} q(x))\Gamma^{\mu\nu\alpha\beta}_{3/2} q(y)\,. \tag{18}$$

Among them $N_{20\mu\nu}$ vanishes due to the Pauli principle. Here $\Gamma^{\mu\nu\alpha\beta}$ is another $J = \frac{3}{2}$ projection operator defined as

$$\Gamma^{\mu\nu\alpha\beta} = \left(g^{\mu\alpha}g^{\nu\beta} - \frac{1}{2}g^{\nu\beta}\gamma^\mu\gamma^\alpha + \frac{1}{2}g^{\mu\beta}\gamma^\nu\gamma^\alpha + \frac{1}{6}\sigma^{\mu\nu}\sigma^{\alpha\beta}\right), \tag{19}$$

The Fierz transformation provides the following relations

$$\begin{pmatrix} N_{10\mu\nu} \\ N_{5\mu\nu} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} N_{20\mu\nu} \\ N_{15\mu\nu} \end{pmatrix}, \tag{20}$$

and

$$\begin{pmatrix} N_{20\mu\nu} \\ N_{15\mu\nu} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} N_{10\mu\nu} \\ N_{5\mu\nu} \end{pmatrix}. \tag{21}$$

Solving these equations, we obtain

$$N_{10\mu\nu} = -N_{5\mu\nu}\,, N_{15\mu\nu} = -2N_{5\mu\nu}\,. \tag{22}$$

Thus, we take $N_{5\mu\nu}$ as the independent field.

We have, therefore, the grand total of four independent bi-local baryon fields $(N_{3\mu}, N_{4\mu}, N_{5\mu}, N_{5\mu\nu})$ with $J = \frac{3}{2}$ and $I = \frac{1}{2}$.

2.5 $J = \frac{3}{2}$ and $I = \frac{3}{2}$ fields

There are two possible fields/Lorentz group representations with $J = \frac{3}{2}$: 1) the $D(1, \frac{1}{2})$ and 2) the $D(\frac{3}{2}, 0)$.

2.5.1 $D(1, \frac{1}{2})$ and $I = \frac{3}{2}$

For $D(1, \frac{1}{2})_{I=\frac{3}{2}}$, there are six operators

$$\begin{cases} \Delta^i_{4\mu}(x,y) = (\tilde{q}(x)\gamma_\nu\gamma_5\tau^j q(y))\Gamma^{\mu\nu}_{3/2}P^{ij}_{3/2}q(x)\,, \\ \Delta^i_{5\mu}(x,y) = (\tilde{q}(x)\sigma_{\alpha\beta}\tau^j q(y))\Gamma^{\mu\alpha}_{3/2}\gamma^\beta\gamma_5 P^{ij}_{3/2}q(x)\,, \end{cases} \tag{23}$$

$$\Delta^i_{8\mu}(x,y) = (\tilde{q}(x)\gamma_\nu\tau^j q(y))\Gamma^{\mu\nu}_{3/2}\gamma_5 P^{ij}_{3/2}q(x)\,, \tag{24}$$

$$\begin{cases} \Delta^i_{14\mu}(x,y) = (\tilde{q}(x)\gamma_\nu\gamma_5\tau^j q(x))\Gamma^{\mu\nu}_{3/2}P^{ij}_{3/2}q(y)\,, \\ \Delta^i_{15\mu}(x,y) = (\tilde{q}(x)\sigma_{\alpha\beta}\tau^j q(x))\Gamma^{\mu\alpha}_{3/2}\gamma^\beta\gamma_5 P^{ij}_{3/2}q(y)\,, \end{cases} \tag{25}$$

$$\Delta^i_{18\mu}(x,y) = (\tilde{q}(x)\gamma_\nu\tau^i q(x))\Gamma^{\mu\nu}_{3/2}\gamma_5 P^{ij}_{3/2}q(y)\,. \tag{26}$$

Among them $\Delta^i_{18\mu}$ vanishes due to the Pauli principle. The Fierz transformation provides the following relations

$$\begin{pmatrix} \Delta^i_{8\mu} \\ \Delta^i_{4\mu} \\ \Delta^i_{5\mu} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix}\begin{pmatrix} \Delta^i_{18\mu} \\ \Delta^i_{14\mu} \\ \Delta^i_{15\mu} \end{pmatrix}, \tag{27}$$

and

$$\begin{pmatrix} \Delta^i_{18\mu} \\ \Delta^i_{14\mu} \\ \Delta^i_{15\mu} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{pmatrix}\begin{pmatrix} \Delta^i_{8\mu} \\ \Delta^i_{4\mu} \\ \Delta^i_{5\mu} \end{pmatrix}. \tag{28}$$

Solving these equations, we obtain

$$\begin{pmatrix} \Delta^i_{8\mu} \\ \Delta^i_{14\mu} \\ \Delta^i_{15\mu} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} \Delta^i_{4\mu} \\ \Delta^i_{5\mu} \end{pmatrix}, \tag{29}$$

i.e. there are two independent Δ fields: $\Delta^i_{4\mu}, \Delta^i_{5\mu}$.

2.5.2 $D(\frac{3}{2}, 0)$ and $I = \frac{3}{2}$

Finally in the $D(\frac{3}{2}, 0)_{I=\frac{3}{2}}$ Lorentz representation, there are only two Δ operators

$$\begin{aligned} \Delta^i_{5\mu\nu}(x,y) &= (\tilde{q}(x)\sigma_{\alpha\beta}\tau^j q(y))\Gamma^{\mu\nu\alpha\beta}_{3/2}P^{ij}_{3/2}q(x)\,, \\ \Delta^i_{15\mu\nu}(x,y) &= (\tilde{q}(x)\sigma_{\alpha\beta}\tau^j q(x))\Gamma^{\mu\nu\alpha\beta}_{3/2}P^{ij}_{3/2}q(y)\,. \end{aligned} \tag{30}$$

The Fierz transformation provides the following relation

$$\Delta^i_{5\mu\nu} = \Delta^i_{15\mu\nu}\,. \tag{31}$$

i.e. there is one independent Δ field: $\Delta^i_{5\mu\nu}$.

We have, therefore, the grand total of three independent bi-local Δ baryon fields, $(\Delta^i_{4\mu}, \Delta^i_{5\mu}, \Delta^i_{5\mu\nu})$ with $J = \frac{3}{2}$ and $I = \frac{3}{2}$.

3 Chiral Transformations

In this section, we investigate the chiral transformations of bi-local baryon operators. The chiral mixing of baryon operators is caused by their diquark components, so it is convenient to classify the baryon operators according to their diquark chiral multiplets: $D_1, D_2 \in (0,\ 0)$, $D_3^\mu, D_4^{\mu i} \in (\frac{1}{2},\ \frac{1}{2})$ and $D_5^{\mu\nu i} \in (1,\ 0) + (0,\ 1)$. The analysis in the section is similar to our previous papers [9,11] about local fields, so we simply list the results.

3.1 $J = \frac{1}{2}$

Under the Abelian chiral transformation the rule, we have

$$\begin{aligned} \delta_5 N_1 &= ia\gamma_5(N_1 + 2N_2)\,, & (32)\\ \delta_5 N_2 &= ia\gamma_5(2N_1 + N_2)\,, & (33)\\ \delta_5 N_3 &= -ia\gamma_5 N_3\,, & (34)\\ \delta_5 N_4 &= -ia\gamma_5 N_4\,, & (35)\\ \delta_5 N_5 &= 3a\gamma_5 N_5\,, & (36) \end{aligned}$$

and

$$\begin{aligned} \delta_5 \Delta_4^i &= -ia\gamma_5 \Delta_4^i\,, & (37)\\ \delta_5 \Delta_5^i &= 3ia\gamma_5 \Delta_5^i\,. & (38) \end{aligned}$$

We can diagonalize N_1 and N_2 and obtain

$$\begin{aligned} \delta_5(N_1 + N_2) &= 3ia\gamma_5(N_1 + N_2)\,, & (39)\\ \delta_5(N_1 - N_2) &= -ia\gamma_5(N_1 - N_2)\,. & (40) \end{aligned}$$

Under the $SU(2)_A$ chiral transformation the rule, we have

$$\begin{aligned} \delta_5^{\boldsymbol{a}} N_1 &= i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5 N_1\,, & (41)\\ \delta_5 N_2 &= i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5 N_2\,, & (42)\\ \delta_5^{\boldsymbol{a}} N_3 &= -i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5 N_3 - \frac{2}{3}i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5 N_4 - 2i\gamma_5\boldsymbol{a}\cdot\boldsymbol{\Delta}_4\,, & (43)\\ \delta_5^{\boldsymbol{a}} N_4 &= -2i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5 N_3 + \frac{1}{3}i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5 N_4 - 2i\gamma_5\boldsymbol{a}\cdot\boldsymbol{\Delta}_4\,, & (44)\\ \delta_5^{\boldsymbol{a}} N_5 &= i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5 N_5\,, & (45) \end{aligned}$$

and

$$\begin{aligned} \delta_5^{\boldsymbol{a}} \Delta_4^i &= -2i\gamma_5 a^j P_{3/2}^{ij} N_3 - \frac{2}{3}i\gamma_5 a^j P_{3/2}^{ij} N_4 + \frac{2}{3}i\tau^i\gamma_5\boldsymbol{a}\cdot\boldsymbol{\Delta}_4 - i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5\Delta_4^i\,, & (46)\\ \delta_5^{\boldsymbol{a}} \Delta_5^i &= -2i\tau^i\gamma_5\boldsymbol{a}\cdot\boldsymbol{\Delta}_5 + 3i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5\Delta_5^i\,. \end{aligned}$$

We find that N_3, N_4 and Δ_4^i can be reduced to irreducible components by taking the antisymmetric linear combination of the two nucleon fields:

$$\begin{aligned} \delta_5^{\boldsymbol{a}}(N_3 - N_4) &= i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5(N_3 - N_4)\,, & (47)\\ \delta_5^{\boldsymbol{a}}(3N_3 + N_4) &= -i\gamma_5\left[\frac{5}{3}\boldsymbol{a}\cdot\boldsymbol{\tau}(3N_3 + N_4) + 8\boldsymbol{a}\cdot\boldsymbol{\Delta}_4\right]\,, & (48)\\ \delta_5^{\boldsymbol{a}}\Delta_4^i &= -i\gamma_5\Big[\frac{2}{3}a^j P_{3/2}^{ij}(3N_3 + N_4) - \frac{2}{3}\tau^i\boldsymbol{a}\cdot\boldsymbol{\Delta}_4 + \boldsymbol{a}\cdot\boldsymbol{\tau}\Delta_4^i\Big]\,. & (49) \end{aligned}$$

Thus we find that $(N_1 \pm N_2)$, $(N_3 - N_4)$, and N_5 form four independent $(\frac{1}{2}, 0)$ chiral multiplets, $\big(N_3 + \frac{1}{3}N_4, \Delta_4^i\big)$ form one $(1, \frac{1}{2})$ chiral multiplet and Δ_5^i forms one $(\frac{3}{2}, 0)$ chiral multiplet.

3.2 $J = \frac{3}{2}$

Under the Abelian chiral transformation rule, we have

$$\delta_5 N_{3\mu} = ia\gamma_5 N_{3\mu} \,, \tag{50}$$
$$\delta_5 N_{4\mu} = ia\gamma_5 N_{4\mu} \,, \tag{51}$$
$$\delta_5 N_{5\mu} = ia\gamma_5 N_{5\mu} \,, \tag{52}$$

and

$$\delta_5 \Delta^i_{4\mu} = ia\gamma_5 \Delta^i_{4\mu} \,, \tag{53}$$
$$\delta_5 \Delta^i_{5\mu} = ia\gamma_5 \Delta^i_{5\mu} \,. \tag{54}$$

$$\delta_5 N_{5\mu\nu} = 3ia\gamma_5 N_{5\mu\nu} \,, \tag{55}$$
$$\delta_5 \Delta^i_{5\mu\nu} = 3ia\gamma_5 \Delta^i_{5\mu\nu} \,. \tag{56}$$

Under the $SU(2)_A$ chiral transformation rule, we have

$$\delta_5^{\boldsymbol{a}} N_{3\mu} = i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5 N_{3\mu} + \frac{2}{3} i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5 N_{4\mu} + 2i\gamma_5 \boldsymbol{a}\cdot\boldsymbol{\Delta}^\mu_4 \,, \tag{57}$$
$$\delta_5^{\boldsymbol{a}} N_{4\mu} = 2i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5 N_{3\mu} - \frac{1}{3} i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5 N_{4\mu} + 2i\gamma_5 \boldsymbol{a}\cdot\boldsymbol{\Delta}^\mu_4 \,, \tag{58}$$
$$\delta_5^{\boldsymbol{a}} N_{5\mu} = \frac{5}{3} i\boldsymbol{\tau}\cdot\boldsymbol{a}\gamma_5 N_{5\mu} - 4i\gamma_5 \boldsymbol{a}\cdot\boldsymbol{\Delta}_{5\mu} \,, \tag{59}$$
$$\delta_5^{\boldsymbol{a}} \Delta^{\mu i}_4 = 2i\gamma_5 a^j P^{ij}_{\frac{3}{2}} N_{3\mu} + \frac{2}{3} i\gamma_5 a^j P^{ij}_{\frac{3}{2}} N_{4\mu} - \frac{2}{3} i\tau^i\gamma_5 \boldsymbol{a}\cdot\boldsymbol{\Delta}^\mu_4 + i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5 \Delta^{\mu i}_4 \,, \tag{60}$$
$$\delta_5^{\boldsymbol{a}} \Delta^{\mu i}_5 = -\frac{4}{3} i\gamma_5 a^j P^{ij}_{3/2} N^\mu_5 - \frac{2}{3} i\tau^i\gamma_5 \boldsymbol{a}\cdot\boldsymbol{\Delta}^\mu_5 + i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5 \Delta^{\mu i}_5 \,, \tag{61}$$

and

$$\begin{aligned} \delta_5^{\boldsymbol{a}} N_{5\mu\nu} &= i\boldsymbol{\tau}\cdot\boldsymbol{a}\gamma_5 N_{5\mu\nu} \,, \\ \delta_5^{\boldsymbol{a}} \Delta^i_{5\mu\nu} &= -2i\tau^i\gamma_5 \boldsymbol{a}\cdot\boldsymbol{\Delta}_{5\mu\nu} + 3i\boldsymbol{a}\cdot\boldsymbol{\tau}\gamma_5 \Delta^i_{5\mu\nu} \,. \end{aligned} \tag{62}$$

Thus we find that $(N_{3\mu} - N_{4\mu})$ form one $(\frac{1}{2}, 0)$ chiral multiplet; $\left(N_{3\mu} + \frac{1}{3}N_{4\mu}, \Delta^i_{4\mu}\right)$ and $\left(N_{5\mu}, \Delta^i_{5\mu}\right)$ form two independent $(1, \frac{1}{2})$ chiral multiplets; $N_{5\mu\nu} \in (\frac{1}{2}, 0)$, $\Delta^i_{5\mu\nu} \in (\frac{3}{2}, 0)$ are also independent chiral multiplets.

4 Summary and Conclusions

We have investigated the chiral multiplets consisting of bi-local three-quark baryon operators, where we took into account the Pauli principle by way of the Fierz transformation. All spin $\frac{1}{2}$ and $\frac{3}{2}$ baryon operators were classified according to their Lorentz and isospin group representations, where spin and isospin projection operators were employed in Tables 1, 2, 3.

We derived the non-trivial relations among various baryon operators due to the Fierz transformations, and thus found the independent baryon fields. Then we found that $(N_1 \pm N_2)$, $(N_3 - N_4)$, and N_5 form four independent $(\frac{1}{2}, 0)$ chiral multiplets, whereas $\left(N_3 + \frac{1}{3}N_4, \Delta^i_4\right)$ form one $(1, \frac{1}{2})$ chiral multiplet and the independent field Δ^i_5 also forms a separate $(\frac{3}{2}, 0)$ chiral multiplet. Thus five nucleons fields, and two Δs, with $J = \frac{1}{2}$ are independent in the bi-local limit, in stark contrast with the local limit where there are two nucleons and no Δ, see Ref. [11].

In the $J = \frac{3}{2}$ sector, the $(N_{3\mu} - N_{4\mu})$ form an independent $(\frac{1}{2}, 0)$ chiral multiplet; $\left(N_{3\mu} + \frac{1}{3}N_{4\mu}, \Delta^i_{4\mu}\right)$ and $\left(N_{5\mu}, \Delta^i_{5\mu}\right)$ form two independent $(1, \frac{1}{2})$ chiral multiplets; $N_{5\mu\nu} \in (\frac{1}{2}, 0)$ and $\Delta^i_{5\mu\nu} \in (\frac{3}{2}, 0)$ are also independent chiral multiplets, again in contrast with the local limit where there is only independent nucleon field and two independent Δ's [11].

This increase of the number of independent fields is in line with our expectations from the non-relativistic quark model, where the number of Pauli-allowed three-quark states in the $L^P = 1^-$ shell sharply rises from the corresponding number in the ground state.

Table 1. The Abelian and the non-Abelian axial charges (+ sign indicates "naive", - sign "mirror" transformation properties) and the non-Abelian chiral multiplets of $J^P = \frac{1}{2}$, Lorentz representation $(\frac{1}{2}, 0)$ nucleon and Δ fields. All of the fields are independent and Fierz invariant.

	$g_A^{(0)}$	$g_A^{(1)}$	$SU_L(2) \times SU_R(2)$
$N_1 - N_2$	-1	$+1$	$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$
$N_1 + N_2$	$+3$	$+1$	$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$
$N_3 - N_4$	-1	$+1$	$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$
$N_3 + \frac{1}{3}N_4$	-1	$-\frac{5}{3}$	$(\frac{1}{2}, 1) \oplus (1, \frac{1}{2})$
Δ_4	-1	$-\frac{1}{3}$	$(\frac{1}{2}, 1) \oplus (1, \frac{1}{2})$
N_5	$+3$	$+1$	$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$
Δ_5	$+3$	$+1$	$(\frac{3}{2}, 0) \oplus (0, \frac{3}{2})$

Table 2. The Abelian and the non-Abelian axial charges and the non-Abelian chiral multiplets of $J^P = \frac{3}{2}$, Lorentz representation $(1, \frac{1}{2})$ nucleon and Δ fields. All of the fields are independent and Fierz invariant.

	$g_A^{(0)}$	$g_A^{(1)}$	$SU_L(2) \times SU_R(2)$
$N_3^\mu - N_4^\mu$	$+1$	-1	$(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$
$N_3^\mu + \frac{1}{3}N_4^\mu$	$+1$	$+\frac{5}{3}$	$(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$
Δ_4^μ	$+1$	$+\frac{1}{3}$	$(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$
N_5^μ	$+1$	$+\frac{5}{3}$	$(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$
Δ_5^μ	$+1$	$+\frac{1}{3}$	$(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$

Table 3. The Abelian and the non-Abelian axial charges and the non-Abelian chiral multiplets of $J = \frac{3}{2}$, Lorentz representation $(\frac{3}{2}, 0)$ nucleon and Δ fields. All of the fields are independent and Fierz invariant.

	$U_A(1)$	$SU_A(2)$	$SU_V(2) \times SU_A(2)$
$N_5^{\mu\nu}$	$+3$	$+1$	$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$
$\Delta_5^{\mu\nu}$	$+3$	$+1$	$(\frac{3}{2}, 0) \oplus (0, \frac{3}{2})$

As in the case of local operators, we showed that the Fierz transformation connects only the baryon operators with identical group-theoretical properties, i.e., belonging to the same chiral multiplet. Then we studied chiral transformations of the bi-local baryon operators. We found that baryons with different isospins may mix under the chiral transformations, i.e., they may belong to the same chiral multiplet. The parity does not play an apparent role in the chiral properties of the baryon operators at this (non-dynamical) level.

One of potential applications of our results should be in attempts to determine the baryons' chiral mixing coefficients/angles, such as Refs. [12,13,7,14]. This is by no means straightforward business, as there is no guarantee that these angles are observables. In the case of the ground state one is fortunate to have the flavor-singlet and octet axial couplings as an external input into the mixing formalism that leads to satisfactory fits to baryon/hyperon masses with reasonable subsequent physical conclusions [7,14].

The framework presented here holds in standard approaches to QCD, such as lattice QCD and the QCD sum rules, under the proviso that chiral symmetry is observed by the approximation used. There is another (sub-)field where it ought to make an impact: on the class of fully relativistic approaches, such as those based on the Bethe-Salpeter equation, to chiral quark models [15,16,17,18,19,20,21].

We have employed the standard isospin formalism instead of the explicit expressions in terms of different flavored quarks in the flavor components of the baryon fields that are commonplace in this line of work. By using the isospin formalism, we have been able to derive all Fierz identities and chiral transformations of the baryons systematically. The extension to $SU(3)$ is not as straightforward as one might have imagined, however, so we leave it for another occasion.

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A Bi-local Diquarks

We begin with bi-linears of two quarks in Eq. (1). There are ten non-vanishing possibilities for Γ_1 with bi-local fields: besides the familiar five (that survive in the local approximation limit)

$$D_1(x,y) = \tilde{q}(x)q(y), \tag{63}$$
$$D_2(x,y) = \tilde{q}(x)\gamma^5 q(y), \tag{64}$$
$$D_3^{\mu}(x,y) = \tilde{q}(x)\gamma^{\mu} q(y), \tag{65}$$
$$D_4^{\mu i}(x,y) = \tilde{q}(x)\gamma^{\mu}\gamma^5\tau^i q(y), \tag{66}$$
$$D_5^{\mu\nu i}(x,y) = \tilde{q}(x)\sigma^{\mu\nu}\tau^i q(y). \tag{67}$$

there are also five new ones (that exist only in the non-local case):

$$D_6^{i}(x,y) = \tilde{q}(x)\tau^i q(y), \tag{68}$$
$$D_7^{i}(x,y) = \tilde{q}(x)\gamma^5\tau^i q(y), \tag{69}$$
$$D_8^{\mu i}(x,y) = \tilde{q}(x)\gamma^{\mu}\tau^i q(y), \tag{70}$$
$$D_9^{\mu}(x,y) = \tilde{q}(x)\gamma^{\mu}\gamma^5 q(y), \tag{71}$$
$$D_{10}^{\mu\nu}(x,y) = \tilde{q}(x)\sigma^{\mu\nu} q(y). \tag{72}$$

These quark bi-linear fields, D_1, D_6^i, D_2, D_7^i, $D_3^{\mu}, D_8^{\mu i}$, $D_4^{\mu i}, D_9^{\mu}$ and $D_5^{\mu\nu i}, D_{10}^{\mu\nu}$, shall be referred to as the isoscalar or isovector, depending on their isospin dependence, scalar, pseudo-scalar, vector, axial-vector and tensor diquarks, according to their Lorentz transformation properties [2].

A.1 Taylor expansion and the Pauli principle

Next we make the Taylor expansion and keep terms through the first order (we define $\overline{x}_{\mu} = \frac{1}{2}(x+y)_{\mu}$; $\Delta_{\mu} = (x-y)_{\mu}$):

$$D_1(x,y) = \tilde{q}(\overline{x})q(\overline{x}) - \frac{1}{2}\Delta^{\alpha}\tilde{q}(x)(\overrightarrow{\partial}^x_{\alpha} - \overleftarrow{\partial}^x_{\alpha})q(x) + \ldots, \tag{73}$$
$$D_2(x,y) = \tilde{q}(\overline{x})\gamma^5 q(\overline{x}) - \frac{1}{2}\Delta^{\alpha}\tilde{q}(x)\gamma^5(\overrightarrow{\partial}^x_{\alpha} - \overleftarrow{\partial}^x_{\alpha})q(x) + \ldots, \tag{74}$$
$$D_3^{\mu}(x,y) = \tilde{q}(\overline{x})\gamma^{\mu} q(\overline{x}) - \frac{1}{2}\Delta^{\alpha}\tilde{q}(x)\gamma^{\mu}(\overrightarrow{\partial}^x_{\alpha} - \overleftarrow{\partial}^x_{\alpha})q(x) + \ldots, \tag{75}$$
$$D_4^{\mu i}(x,y) = \tilde{q}(\overline{x})\gamma^{\mu}\gamma^5\tau^i q(\overline{x}) - \frac{1}{2}\Delta^{\alpha}\tilde{q}(x)\gamma^{\mu}\gamma^5\tau^i(\overrightarrow{\partial}^x_{\alpha} - \overleftarrow{\partial}^x_{\alpha})q(x) + \ldots, \tag{76}$$
$$D_5^{\mu\nu i}(x,y) = \tilde{q}(\overline{x})\sigma^{\mu\nu}\tau^i q(\overline{x}) - \frac{1}{2}\Delta^{\alpha}\tilde{q}(x)\sigma^{\mu\nu}\tau^i(\overrightarrow{\partial}^x_{\alpha} - \overleftarrow{\partial}^x_{\alpha})q(x) + \ldots. \tag{77}$$

and

$$D_6^{i}(x,y) = \tilde{q}(\overline{x})\tau^i q(\overline{x}) - \frac{1}{2}\Delta^{\alpha}\tilde{q}(x)(\overrightarrow{\partial}^x_{\alpha} - \overleftarrow{\partial}^x_{\alpha})\tau^i q(x) + \ldots, \tag{78}$$
$$D_7^{i}(x,y) = \tilde{q}(\overline{x})\gamma^5\tau^i q(\overline{x}) - \frac{1}{2}\Delta^{\alpha}\tilde{q}(x)(\overrightarrow{\partial}^x_{\alpha} - \overleftarrow{\partial}^x_{\alpha})\tau^i q(x) + \ldots, \tag{79}$$
$$D_8^{\mu i}(x,y) = \tilde{q}(\overline{x})\gamma^{\mu}\tau^i q(\overline{x}) - \frac{1}{2}\Delta^{\alpha}\tilde{q}(x)\gamma^{\mu}(\overrightarrow{\partial}^x_{\alpha} - \overleftarrow{\partial}^x_{\alpha})\tau^i q(x) + \ldots, \tag{80}$$
$$D_9^{\mu}(x,y) = \tilde{q}(\overline{x})\gamma^{\mu}\gamma^5 q(\overline{x}) - \frac{1}{2}\Delta^{\alpha}\tilde{q}(x)\gamma^{\mu}\gamma^5(\overrightarrow{\partial}^x_{\alpha} - \overleftarrow{\partial}^x_{\alpha})q(x) + \ldots, \tag{81}$$
$$D_{10}^{\mu\nu}(x,y) = \tilde{q}(\overline{x})\sigma^{\mu\nu} q(\overline{x}) - \frac{1}{2}\Delta^{\alpha}\tilde{q}(x)\sigma^{\mu\nu}(\overrightarrow{\partial}^x_{\alpha} - \overleftarrow{\partial}^x_{\alpha})q(x) + \ldots. \tag{82}$$

Taking into account the Pauli principle, we find

$$D_1(x,y) = \tilde{q}(\overline{x})q(\overline{x}) + \mathcal{O}(\Delta^2), \tag{83}$$

[2] Throughout the present paper, Latin indices i, j etc. run over the isospin space 1, 2, 3, and Greek indices α, β etc. run over the Lorentz space 0, 1, 2, 3.

$$D_2(x,y) = \tilde{q}(\overline{x})\gamma^5 q(\overline{x}) + \mathcal{O}(\Delta^2), \tag{84}$$
$$D_3^{\mu}(x,y) = \tilde{q}(\overline{x})\gamma^{\mu} q(\overline{x}) + \mathcal{O}(\Delta^2), \tag{85}$$
$$D_4^{\mu i}(x,y) = \tilde{q}(\overline{x})\gamma^{\mu}\gamma^5\tau^i q(\overline{x}) + \mathcal{O}(\Delta^2), \tag{86}$$
$$D_5^{\mu\nu i}(x,y) = \tilde{q}(\overline{x})\sigma^{\mu\nu}\tau^i q(\overline{x}) + \mathcal{O}(\Delta^2). \tag{87}$$

and

$$D_6^{i}(x,y) = -\frac{1}{2}\Delta^{\alpha}\tilde{q}(x)(\overrightarrow{\partial}^x_{\alpha} - \overleftarrow{\partial}^x_{\alpha})\tau^i q(x) + \dots, \tag{88}$$
$$D_7^{i}(x,y) = -\frac{1}{2}\Delta^{\alpha}\tilde{q}(x)(\overrightarrow{\partial}^x_{\alpha} - \overleftarrow{\partial}^x_{\alpha})\tau^i q(x) + \dots, \tag{89}$$
$$D_8^{\mu i}(x,y) = -\frac{1}{2}\Delta^{\alpha}\tilde{q}(x)\gamma^{\mu}(\overrightarrow{\partial}^x_{\alpha} - \overleftarrow{\partial}^x_{\alpha})\tau^i q(x) + \dots, \tag{90}$$
$$D_9^{\mu}(x,y) = -\frac{1}{2}\Delta^{\alpha}\tilde{q}(x)\gamma^{\mu}\gamma^5(\overrightarrow{\partial}^x_{\alpha} - \overleftarrow{\partial}^x_{\alpha}) q(x) + \dots, \tag{91}$$
$$D_{10}^{\mu\nu}(x,y) = -\frac{1}{2}\Delta^{\alpha}\tilde{q}(x)\sigma^{\mu\nu}(\overrightarrow{\partial}^x_{\alpha} - \overleftarrow{\partial}^x_{\alpha}) q(x) + \dots. \tag{92}$$

A.2 Chiral properties of diquarks

Firstly we look at their Abelian chiral ($U(1)_A$) transformation properties, which are determined by the $U(1)_A$ transformation of the quark field,

$$q \to \exp(i\gamma_5 a) q = q + \delta_5 q, \tag{93}$$
$$\tilde{q} \to \tilde{q}\exp(i\gamma_5 a) = \tilde{q} + \delta_5 \tilde{q}, \tag{94}$$

where a is an infinitesimal parameter for $U(1)_A$ transformation. The scalar and pseudo-scalar diquarks transform as

$$\delta_5 D_1 = 2ia D_2, \tag{95}$$
$$\delta_5 D_2 = 2ia D_1, \tag{96}$$

from which it follows that

$$\delta_5 (D_1 - D_2) = -2ia (D_1 - D_2), \tag{97}$$
$$\delta_5 (D_1 + D_2) = 2ia (D_1 + D_2), \tag{98}$$

the vector and axial-vector diquarks,

$$\delta_5 D_{3,4} = 0, \tag{99}$$

the tensor diquark,

$$\delta_5 D_5^{\mu\nu i} = 2ia D_{11}^{\mu\nu i}, \tag{100}$$

where $D_{11}^{\mu\nu i} = \tilde{q}\sigma^{\mu\nu}\gamma_5\tau^i q$ is a dual-tensor diquark. Note that baryon operators constructed from the dual-tensor diquark can be expressed as functions of the tensor diquark by using the identity $\sigma^{\mu\nu}\gamma_5 = -\frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}$.

Next we consider the axial transformation

$$q \to \exp(i\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{a}) q = q + \delta_5^{\boldsymbol{a}} q, \tag{101}$$
$$\tilde{q} \to \tilde{q}\exp(-i\boldsymbol{\tau}\cdot\boldsymbol{a}\gamma_5) = \tilde{q} + \delta_5^{\boldsymbol{a}} \tilde{q}, \tag{102}$$

where $\boldsymbol{a}$ is the triplet for the axial transformation parameters. It is straightforward to obtain the axial transformations of the diquarks: for the scalar and pseudo-scalar diquarks,

$$\delta_5^{\boldsymbol{a}} D = 0,\ (D = D_1, D_2), \tag{103}$$

for the vector and axial-vector diquarks,

$$\delta_5^{\boldsymbol{a}} D_3^{\mu} = 2ia^i D_4^{\mu i}, \tag{104}$$
$$\delta_5^{\boldsymbol{a}} D_4^{\mu i} = 2ia^i D_3^{\mu}, \tag{105}$$

for the tensor diquark,

$$\delta_5^{\boldsymbol{a}} D_5^{\mu\nu i} = -2\epsilon^{ijk} a^j D_{11}^{\mu\nu k}. \tag{106}$$

These transformations imply that the scalar and pseudo-scalar diquarks $D_{1,2}$ are chiral scalars (invariants) (0, 0). The vector and axial-vector diquarks $D_3^\mu,\ D_4^{\mu i}$ together belong to the chiral multiplet $(\frac{1}{2},\ \frac{1}{2})$; therefore they are chiral partners, similar to the $(\sigma,\ \boldsymbol{\pi})$ case. The tensor diquark transforms into the dual-tensor diquark under non-Abelian chiral transformations, and the two together form the chiral multiplet $(1,0)\oplus(0,1)$.

Similarly, the "Pauli forbidden" diquarks transform as follows under the $U_A(1)$ axial transformation.

The scalar and pseudo-scalar diquarks transform as

$$\delta_5 D_6 = 2iaD_7, \tag{107}$$
$$\delta_5 D_7 = 2iaD_6, \tag{108}$$

from which follows

$$\delta_5(D_6 - D_7) = -2ia(D_6 - D_7), \tag{109}$$
$$\delta_5(D_6 + D_7) = 2ia(D_6 + D_7), \tag{110}$$

the vector and axial-vector diquarks,

$$\delta_5 D_{8,9} = 0, \tag{111}$$

the tensor diquark,

$$\delta_5 D_{10}^{\mu\nu} = 2iaD_{11}^{\mu\nu}, \tag{112}$$

where $D_{11}^{\mu\nu} = \tilde{q}\sigma^{\mu\nu}\gamma_5 q$ is a dual-tensor diquark. Note that baryon operators constructed from the dual-tensor diquark can be expressed as functions of the tensor diquark by using the identity $\sigma^{\mu\nu}\gamma_5 = -\frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}$.

Next we consider the non-Abelian axial transformation, Eqs.(102). It is straightforward to obtain the axial transformations of the diquarks: for the scalar and pseudo-scalar diquarks,

$$\begin{aligned}\delta_5^{\boldsymbol{a}} D_6^i &= -2\epsilon^{ijk} a^j D_7^k,\\ \delta_5^{\boldsymbol{a}} D_7^i &= -2\epsilon^{ijk} a^j D_6^k,\end{aligned} \tag{113}$$

for the vector and axial-vector diquarks,

$$\delta_5^{\boldsymbol{a}} D_9^\mu = 2ia^i D_8^{\mu i}, \tag{114}$$
$$\delta_5^{\boldsymbol{a}} D_8^{\mu i} = 2ia^i D_9^\mu, \tag{115}$$

for the tensor diquark,

$$\delta_5^{\boldsymbol{a}} D_{10}^{\mu\nu} = 0. \tag{116}$$

These transformations imply that the scalar and pseudo-scalar diquarks D_6^i, D_7^i together form the chiral multiplet $(1,0)\oplus(0,1)$, they are chiral partners, similar to the $(\boldsymbol{\rho},\ \boldsymbol{a_1})$ mesons. The vector and axial-vector diquarks $D_3^\mu,\ D_4^{\mu i}$ together belong to the chiral multiplet $(\frac{1}{2},\ \frac{1}{2})$; therefore they are chiral partners, similar to the $(\sigma,\ \boldsymbol{\pi})$ case. The isoscalar tensor diquark transforms into the isoscalar dual-tensor diquark under non-Abelian chiral transformations, so they are chiral scalars (invariants) (0, 0).

All spin 0 and 1 diquark operators were classified according to their Lorentz and isospin group representations, in Table 4. From the above Table 4 we see that only the Lorentz rep. $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ diquark fields have the same chiral properties with both signs under the Pauli principle/two particle exchange. This means that only the baryon fields made from these diquarks are subject to "Pauli mixing". These are the $J=\frac{1}{2}$ and $J=\frac{3}{2}$ fields.

B Dirac, Isospin and Spatial Fierz Transformations

We summarize the detailed results of the Fierz transformation of baryons. After the combined Fierz transformations of the iso-spin, Dirac, spatial and color degrees of freedom, the Fierz transformed field $\mathcal{F}[N]$ must satisfy the relation $N = -\mathcal{F}[N]$, namely the Pauli principle.

In order to derive the following results, it is useful to have the following Fierz identities.

Table 4. The Abelian $U_A(1)$ and the non-Abelian $SU_L(2)\times SU_R(2)$ chiral transformation properties/axial charges and the Lorentz group representations of the diquark fields. In the last column we show the sign under transposition of the two quarks in the colour $\overline{3}$ state.

	$U_A(1)$	Lorentz group $SO(3,1)$	$SU_L(2)\times SU_R(2)$	exchange
D_1-D_2	-1	$(0,0)$	$(0,0)$	$+$
D_1+D_2	$+1$	$(0,0)$	$(0,0)$	$+$
D_3	0	$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$+$
D_4	0	$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$+$
D_5	$+1$	$(1,0)\oplus(0,1)$	$(1,0)\oplus(0,1)$	$+$
D_6-D_7	-1	$(0,0)$	$(1,0)\oplus(0,1)$	$-$
D_6+D_7	$+1$	$(0,0)$	$(1,0)\oplus(0,1)$	$-$
D_8	0	$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$-$
D_9	0	$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$-$
D_{10}	$+1$	$(1,0)\oplus(0,1)$	$(0,0)$	$-$

B.1 Isospin Fierz Transformations

• Isospin:

$$\begin{pmatrix}(\tau_0)_{12}(\tau_0)_{34}\\(\tau_i)_{12}(\tau_i)_{34}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1\\3&-1\end{pmatrix}\begin{pmatrix}(\tau_0)_{14}(\tau_0)_{23}\\(\tau_i)_{14}(\tau_i)_{23}\end{pmatrix} \tag{117}$$

$$(\tau^j)_{12}(P^{ij}_{3/2})_{34} = (\tau^j)_{14}(P^{ij}_{3/2})_{32}. \tag{118}$$

B.2 Dirac Fierz Transformations

• Spin:

$$\begin{pmatrix}(1)_{12}(1)_{34}\\(\gamma_5)_{12}(\gamma_5)_{34}\\(\gamma^\mu)_{12}(\gamma_\mu)_{34}\\(\gamma^\mu\gamma_5)_{12}(\gamma_\mu\gamma_5)_{34}\\(\sigma^{\mu\nu})_{12}(\sigma_{\mu\nu})_{34}\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1&1&1&-1&\frac{1}{2}\\1&1&-1&1&\frac{1}{2}\\4&-4&-2&-2&0\\-4&4&-2&-2&0\\12&12&0&0&-2\end{pmatrix}\begin{pmatrix}(1)_{14}(1)_{32}\\(\gamma_5)_{14}(\gamma_5)_{32}\\(\gamma^\mu)_{14}(\gamma_\mu)_{32}\\(\gamma^\mu\gamma_5)_{14}(\gamma_\mu\gamma_5)_{32}\\(\sigma^{\mu\nu})_{14}(\sigma_{\mu\nu})_{32}\end{pmatrix}, \tag{119}$$

$$\begin{pmatrix}(\gamma_\nu)_{12}(\Gamma^{\mu\nu}_{3/2})_{34}\\(\gamma_\nu\gamma_5)_{12}(\Gamma^{\mu\nu}_{3/2}\gamma_5)_{34}\\i(\sigma_{\beta\alpha})_{12}(\Gamma^{\mu\beta}_{3/2}\gamma^\alpha)_{34}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1&-1\\1&1&1\\-2&2&0\end{pmatrix}\begin{pmatrix}(\gamma_\nu)_{14}(\Gamma^{\mu\nu}_{3/2})_{32}\\(\gamma_\nu\gamma_5)_{14}(\Gamma^{\mu\nu}_{3/2}\gamma_5)_{32}\\i(\sigma_{\beta\alpha})_{14}(\Gamma^{\mu\beta}_{3/2}\gamma^\alpha)_{32}\end{pmatrix} \tag{120}$$

$$i(\sigma_{\beta\alpha})_{12}(\Gamma^{\mu\beta}_{3/2}\gamma^\alpha)_{34} = -i(\sigma_{\beta\alpha}\gamma_5)_{12}(\Gamma^{\mu\beta}_{3/2}\gamma^\alpha\gamma_5)_{34} \tag{121}$$

$$(\sigma_{\alpha\beta})_{12}(\Gamma^{\mu\nu\alpha\beta}_{3/2})_{34} = (\sigma_{\alpha\beta}\gamma_5)_{12}(\Gamma^{\mu\nu\alpha\beta}_{3/2}\gamma_5)_{34}=(\sigma_{\alpha\beta})_{14}(\Gamma^{\mu\nu\alpha\beta}_{3/2})_{32}. \tag{122}$$

Equations (117) and (119) are well-known Fierz transformations, which are used to obtain spin-$\frac{1}{2}$ and isospin-$\frac{1}{2}$ part of the Fierz identities, the remaining identities Eqs. (118), (120), (122) are necessary to obtain the Fierz transformation of the baryons having spin-$\frac{3}{2}$ and/or isospin-$\frac{3}{2}$.

$$\begin{pmatrix}(\gamma_\nu)_{12}(\Gamma^{\mu\nu}_{3/2}\gamma_5)_{34}\\(\gamma_\nu\gamma_5)_{12}(\Gamma^{\mu\nu}_{3/2})_{34}\\i(\sigma_{\beta\alpha})_{12}(\Gamma^{\mu\beta}_{3/2}\gamma^\alpha\gamma_5)_{34}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1&1\\1&1&-1\\2&-2&0\end{pmatrix}\begin{pmatrix}(\gamma_\nu)_{14}(\Gamma^{\mu\nu}_{3/2}\gamma_5)_{32}\\(\gamma_\nu\gamma_5)_{14}(\Gamma^{\mu\nu}_{3/2})_{32}\\i(\sigma_{\beta\alpha})_{14}(\Gamma^{\mu\beta}_{3/2}\gamma^\alpha\gamma_5)_{32}\end{pmatrix} \tag{123}$$

B.3 Spatial Fierz transformation

Let P_{ij} be the ij-th particle interchange operator and $(\boldsymbol{\rho},\boldsymbol{\lambda})$ the three-body Jacobi coordinates

$$\boldsymbol{\rho}=\frac{1}{\sqrt{2}}(\mathbf{x_1}-\mathbf{x_2}), \tag{124}$$

$$\boldsymbol{\lambda}=\frac{1}{\sqrt{6}}(\mathbf{x_1}+\mathbf{x_2}-2\mathbf{x_3}). \tag{125}$$

Then the three-particle permutation/exchange symmetry s_3 can be examined as

$$P_{12}\boldsymbol{\rho} \to -\boldsymbol{\rho} \tag{126}$$
$$P_{12}\boldsymbol{\lambda} \to \boldsymbol{\lambda} \tag{127}$$
$$P_{13}\boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} - \frac{\sqrt{3}}{2}\boldsymbol{\lambda} \tag{128}$$
$$P_{13}\boldsymbol{\lambda} \to -\frac{\sqrt{3}}{2}\boldsymbol{\rho} - \frac{1}{2}\boldsymbol{\lambda} \tag{129}$$

i.e.,

$$\mathcal{F}\left[\begin{pmatrix}\boldsymbol{\rho}\\ \boldsymbol{\lambda}\end{pmatrix}\right] = P_{23}\left[\begin{pmatrix}\boldsymbol{\rho}\\ \boldsymbol{\lambda}\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\begin{pmatrix}\boldsymbol{\rho}\\ \boldsymbol{\lambda}\end{pmatrix}. \tag{130}$$

References

1. Y. Ohnuki, *Unitary Representations of the Poincare Group and Relativistic Wave Equations*, (World Scientific, Singapore, 1988).
2. B.L. Ioffe, Nucl. Phys. **B 188**, 317 (1981); ibid. **B 191**, 591 (1981) (E); Z. Phys. **C 18**, 67 (1983).
3. Y. Chung, H. G. Dosch, M. Kremer and D. Schall, Nucl. Phys. B **197** (1982) 55.
4. D. Espriu, P. Pascual and R. Tarrach, Nucl. Phys. B **214** (1983) 285.
5. T.D. Cohen and X. Ji, Phys. Rev. **D 55**, 6870 (1997).
6. H. X. Chen, V. Dmitrašinović, A. Hosaka, K. Nagata and S. L. Zhu, Phys. Rev. D **78**, 054021 (2008).
7. H. X. Chen, V. Dmitrasinovic and A. Hosaka, Phys. Rev. D **81**, 054002 (2010) [arXiv:0912.4338 [hep-ph]].
8. S. Basak *et al.* [Lattice Hadron Physics Collaboration (LHPC)], Phys. Rev. D **72**, 074501 (2005) [arXiv:hep-lat/0508018].
9. V. Dmitrašinović, K. Nagata, and A. Hosaka, Mod. Phys. Lett. A **23**, 2381 (2008) [arXiv:0705.1896 [hep-ph]].
10. K. Nagata, A. Hosaka and V. Dmitrašinović, Phys. Rev. Lett. **101**, 092001 (2008) [arXiv:0804.3185 [hep-ph]].
11. K. Nagata, A. Hosaka and V. Dmitrašinović, Eur. Phys. J. C **57** (2008) 557.
12. V. Dmitrašinović, A. Hosaka and K. Nagata, Mod. Phys. Lett. A **25**, 233 (2010) [arXiv:0912.2372 [hep-ph]].
13. V. Dmitrašinović, A. Hosaka and K. Nagata, Int. J. Mod. Phys. E **19**, 91 (2010) [arXiv:0912.2396 [hep-ph]].
14. H. X. Chen, V. Dmitrasinovic and A. Hosaka, arXiv:1009.2422 [hep-ph].
15. A. B. Henriques, B. H. Kellett and R. G. Moorhouse, Annals Phys. **93**, 125 (1975).
16. I. D. King and C. T. Sachrajda, Nucl. Phys. B **279**, 785 (1987).
17. M. G. Sotiropoulos and G. F. Sterman, Nucl. Phys. B **425**, 489 (1994) [arXiv:hep-ph/9401237].
18. N. G. Stefanis, Eur. Phys. J. direct C **7**, 1 (1999) [arXiv:hep-ph/9911375].
19. U. Loring, K. Kretzschmar, B. C. Metsch and H. R. Petry, Eur. Phys. J. A **10**, 309 (2001) [arXiv:hep-ph/0103287].
20. A. Kvinikhidze and G. A. Miller, Phys. Rev. C **76**, 025203 (2007) [arXiv:nucl-th/0701017].
21. G. Eichmann, R. Alkofer, A. Krassnigg and D. Nicmorus, Phys. Rev. Lett. **104**, 201601 (2010) [arXiv:0912.2246 [hep-ph]].